APPROXIMATION OF OSELEDETS SPLITTINGS

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Abstract. We prove that the Oseledets splittings of an ergodic hyperbolic measure of a C^{1+r} diffeomorphism can be approximated by that of atomic measures on hyperbolic periodic orbits. This removes the assumption on simple spectrum in [6] and strengthens Katok's closing lemma.

1. Introduction

Let M be a compact connected d-dimensional Riemannian manifold without boundary. For $1 \le i \le d$, $x \in M$, let $G(i,x)$ denote the i-dimensional Grassmann space of T_xM. Given integers $n_1, n_2 \cdots, n_s$ with $\sum_{i=1}^s n_i = d$ and $1 \le n_s \le d$, we define a bundle $\mathcal{U}(n_1, \cdots, n_s) = \cup_{x\in M}\mathcal{U}(n_1, \cdots, n_s; x)$, where the fiber over x is

$$\mathcal{U}(n_1, \cdots, n_s; x) = \big\{(E_1, \cdots, E_s) \in G(n_1, x) \times \cdots \times G(n_s, x)\big\}.$$

Set $\mathcal{V}(n_1, \cdots, n_s) = \cup_{x\in M}\mathcal{V}(n_1, \cdots, n_s; x)$, where the fiber over x is

$$\mathcal{V}(n_1, \cdots, n_s; x) = \big\{(E_1, \cdots, E_s) \in \mathcal{U}(n_1, \cdots, n_s) \mid E_1(x) \oplus \cdots \oplus E_s(x) = T_xM\big\}.$$

It is immediate that by the definitions, $\mathcal{V}(n_1, \cdots, n_s)$ is not compact but

$$\mathcal{U}(n_1, \cdots, n_s) = \overline{\mathcal{V}(n_1, \cdots, n_s)}$$

is compact.

Given integers $1 \le l_1 < l_2 < \cdots < l_k < d$, we define a filtration bundle $\mathcal{W}(l_1, \cdots, l_k) = \cup_{x\in M} U(l_1, \cdots, l_k; x)$, where the fiber over x is

$$\mathcal{W}(l_1, \cdots, l_k; x) = \big\{(F_1, \cdots, F_k; H_1, \cdots, H_k) \mid \quad F_i \in G(l_i, x),\ H_i \in G(d - l_i, x), \quad F_i \subset F_{i+1}, \quad H_{i+1} \subset H_i\big\}.$$

Clearly, by the definitions, $\mathcal{W}(l_1, \cdots, l_k)$ is compact.

Throughout this paper, we assume the following relations

$$k = s - 1, \quad l_i = \sum_{j=1}^{i} n_j, \quad 1 \le i \le s - 1.$$

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For the sake of statement, we appoint $\mathcal{U}=\mathcal{U}(n_1,\cdots,n_s)$, $\mathcal{V}=\mathcal{V}(n_1,\cdots,n_s)$ and $\mathcal{W}=\mathcal{W}(l_1,\cdots,l_k)$. Given $(E_1,\cdots,E_s)\in\mathcal{V}$, we define

$$\sigma((E_1,\cdots,E_s))=(E_1,E_1\oplus E_2,\cdots,\oplus_{i=1}^{s-1}E_i;\quad \oplus_{i=2}^{s-1}E_i,\cdots,E_{s-1}\oplus E_s,E_s)\in\mathcal{W}.$$

Then $\sigma:\mathcal{V}\to\mathcal{W}$ is a onto diffeomorphism. Denote $\mathcal{W}_1=\sigma(\mathcal{V})$ and $\sigma^{-1}=\sigma^{-1}\mid_{\mathcal{W}_1}$.

For a p-frame $\xi=(u_1,\cdots,u_p)$ which spans a p-dimensional space E, we define $\mathrm{vol}(\xi)=\mathrm{vol}(u_1,u_2,\cdots,u_p)$ the volume of the parallelepiped generated by the vectors $u_1,u_2,\cdots,u_p$. More precisely, we choose an orthonormal $p-$frame $\zeta=(w_1,w_2,\cdots,w_p)$, $w_i\in T_xM,\ \ i=1,\cdots,p$, which generates the same linear subspace of T_xM as ξ does and take a unique $p\times p$ matrix A with $\xi=\zeta A$. Then we define the volume of ξ by

$$\mathrm{vol}(\xi):=|\det A|.$$

We remark that the volume $\mathrm{vol}(\xi)$ does not depend on the choice of ζ, since the determinate of a transition matrix between two orthonormal frames is ± 1. Hence, $\mathrm{vol}(\xi)$ is well defined. Given any linear onto transformation Φ from one linear space E to a linear space F, we define

$$\det(\Phi)=\frac{\mathrm{vol}(\Phi(\xi))}{\mathrm{vol}(\xi)},$$

where ξ is a frame spanning E. The definition does not depend on the choice of ξ.

Let f be a C^1 diffeomorphism of M. For $1\le i\le k$, define projections

$$\pi_i(\gamma)=F_i,\quad \widehat{\pi}_i(\gamma)=H_i,$$

and functions on $\mathcal{W}(l_1,\cdots,l_k)$

$$\phi_i(\alpha)=\det(Df_x\mid_{E_i}),\ \ \psi_i(\gamma)=\det(Df_x\mid_{F_i}),$$

where $\gamma=(F_1,\cdots,F_k;H_1,\cdots,H_k)\in\mathcal{W}(l_1,\cdots,l_k;x)$. Then ϕ_i, ψ_i $(1\le i\le k)$ are bounded and continuous on $\mathcal{W}(l_1,\cdots,l_k)$.

In what follows, suppose f preserves an ergodic measure ω. Then there exist

(a) real numbers $\lambda_1<\cdots<\lambda_s(s\le d)$;
(b) positive integers $n_1,\cdots,n_s$, satisfying $n_1+\cdots+n_s=d$;
(c) a Borel set $O(\omega)$, called Oseledets basin of ω, satisfying $f(O(\omega))=O(\omega)$ and $\omega(O(\omega))=1$;
(d) a measurable splitting, called Oseledets splitting, $T_xM=E_1(x)\oplus\cdots\oplus E_s(x)$ with $\dim E_i(x)=n_i$ and $Df(E_i(x))=E_i(fx)$,

such that

$$\lim_{n\to\pm\infty}\frac{\log\|Df^nv\|}{n}=\lambda_i,$$

for $\forall x\in O(\omega)$, $v\in E_i(x)$, $\ i=1,2,\cdots,s$.

(a) and (d) in the Oseledets theorem allow us to arrange the Oseledets splitting according to the increasing order of the Lyapunov exponents. To avoid excessive terminology, we will arrange the Oseledets splitting at every point in the Oseledets basin in this way throughout this paper without explanation. We call the measure ω hyperbolic if none of its Lyapunov exponents is zero.

From now on, suppose $f:M\to M$ is C^{1+r} diffeomorphism. We give a quick review concerning some notions and results of Pesin theory. We point the reader to [5][15] for more details.

1.1. **Pesin set.**

Definition 1.1. Given $\alpha, \beta \gg \epsilon > 0$, and for all $k \in \mathbb{Z}^+$, the Pesin block $\Lambda_k = \Lambda_k(\alpha, \beta; \epsilon)$ consists of all points $x \in M$ for which there is a splitting $T_xM = E^s_x \oplus E^u_x$ with the invariance property $(Df^m)E^s_x = E^s_{f^m x}$ and $(Df^m)E^u_x = E^u_{f^m x}$, and satisfying:
(a) $\|Df^n|E^s_{f^m x}\| \le e^{\epsilon k} e^{-(\beta-\epsilon)n} e^{\epsilon|m|}, \forall m \in \mathbb{Z}, n \ge 1$;
(b) $\|Df^{-n}|E^u_{f^m x}\| \le e^{\epsilon k} e^{-(\alpha-\epsilon)n} e^{\epsilon|m|}, \forall m \in \mathbb{Z}, n \ge 1$; and
(c) $\tan(Angle(E^s_{f^m x}, E^u_{f^m x})) \ge e^{-\epsilon k} e^{-\epsilon|m|}, \forall m \in \mathbb{Z}$.

Definition 1.2. $\Lambda(\alpha, \beta; \epsilon) = \bigcup_{k=1}^{+\infty} \Lambda_k(\alpha, \beta; \epsilon)$ is a Pesin set.

We say an invariant measure ω is related to the Pesin set $\Lambda = \Lambda(\alpha, \beta; \epsilon)$, if $\omega(\Lambda(\alpha, \beta; \epsilon)) = 1$.

The following theorem is the main theorem in [20].

Theorem 1.3. [20] *Let M be a compact $d-$dimensional Riemannian manifold. Let $f : M \to M$ be a C^{1+r} diffeomorphism, and let ω be an ergodic hyperbolic measure with Lyapunov exponents $\lambda_1 \le \ldots \le \lambda_r < 0 < \lambda_{r+1} \le \ldots \le \lambda_d$. Then the Lyapunov exponents of ω can be approximated by Lyapunov exponents of hyperbolic periodic orbits. More precisely, for any $\varepsilon > 0$, there exists a hyperbolic periodic point z with Lyapunov exponents $\lambda^z_1 \le \ldots \le \lambda^z_d$ such that $\mid \lambda_i - \lambda^z_i \mid < \varepsilon$, $i = 1, ..., d$.*

Definition 1.4. Let E and F be two $Df-$invariant sub-bundles of TM. The angle between $E(x)$ and $F(x)$, $x \in M$, is defined as follows:

$$\sin\angle(E(x),\, F(x)) := \inf_{0 \ne u \in E(x), 0 \ne v \in F(x)} \sin\angle(u,\, v) = \inf_{0 \ne u \in E(x), 0 \ne v \in F(x)} \frac{\|u \wedge v\|}{\|u\|\|v\|},$$

where $\wedge$ denotes the wedge product. We call

$$m\angle(E(x), F(x)) := \lim_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \angle(Df^i E(x),\, Df^i F(x))$$

the mean angle between E and F at x.

We view T_xM as $\mathbb{R}^d$. Given $v \in \mathbb{R}^d$ and a linear subspace $E \subset \mathbb{R}^d$, define

$$\Gamma(v, E) = \begin{cases} \text{the projection of}\, v/\|v\| \;\text{onto}\, E, & v \ne 0, \\ 0, & v = 0. \end{cases}$$

Furthermore, $\Gamma(F, E)$ is viewed as an operator taking values in $F \subset \mathbb{R}^d$. We adopt the classic metric d_G on the Grassmann bundle $\cup_{0 \le i \le d} G(i)$:

$$d_G(F,\, F') = \|\Gamma(\mathbb{R}^l, F) - \Gamma(\mathbb{R}^l, F')\| := \sup_{v \in \mathbb{R}^l} \|\Gamma(v, F) - \Gamma(v, F')\|,$$

where $F, F' \in \cup_{0 \le i \le d} G(i)$. The definitions above can be naturally extended to the translations of linear subspaces. When F, F' are disjoint, d_G is equivalent to the angle $\angle$.

Definition 1.5. Let E and F be two $Df-$invariant sub-bundles of TM. We call

$$md_G(E(x),F(x)) := \lim_{n\to+\infty}\frac{1}{n}\sum_{i=0}^{n-1} d_G(Df^iE(x),\, Df^iF(x))$$

the mean distance between E and F at x.

Suppose that f preserves an ergodic measure ω with its Oseledec splitting

$$T_xM = E_1(x)\oplus\cdots\oplus E_s(x), \quad s\le d=\dim M, \quad x\in O(\omega). \tag{1}$$

By the Birkhoff Ergodic Theorem, there is an $\omega-$full measure subset in $O(\omega)$ such that every point x in this subset satisfies

$$md_G(E_i(x),E_j(x)) = \int d_G(E_i(y),E_j(y))d\omega(y).$$

We call $\int d_G(E_i(y),E_j(y))d\omega(y)$ the mean distance between $E_i := \cup_{x\in O(\omega)}E_i(x)$ and $E_j := \cup_{x\in O(\omega)}E_j(x)$ and write it as $md_G(\omega)(E_i,E_j)$, $\forall\, 1\le i\ne j\le d$.

Given $\gamma=(E_1,\cdots,E_t)\in T_xM\times\cdots\times T_xM$, denote by $A(\gamma)$ the matrix

$$(\det(E_i)\det(E_j)\cos\angle(E_i,E_j))_{t\times t}.$$

Let $\sigma(\gamma)$ denote the set of all eigenvalues of $A(\gamma)$ and let $\tau(\gamma)$ be the smallest eigenvalue. Note that $A(\gamma)$ is a real positive-definite symmetric matrix, therefore, $\sigma(\gamma)\subset(0,+\infty)$.

For $x\in O(\omega)$, we define the independence number of x by the independence number of bundles at x whose elements are all on different invariant bundles. More precisely, take $\gamma(x)=(E_1(x),\, E_2(x),\,\cdots,\, E_s(x))$. Then we define

$$\tau(x) := \tau(\gamma),$$

the smallest eigenvalue of $A(\gamma)$. Clearly, $\tau(x)$ is well defined for $x\in O(\omega)$. Moreover, by the Birkhoff Ergodic Theorem, the equation

$$\lim_{n\to+\infty}\frac{1}{n}\sum_{i=0}^{n-1}\tau(f^ix) = \int\tau(y)d\omega(y)$$

holds on an $\omega-$full measure subset of $O(\omega)$. Therefore, we can define the independence number of ω by

$$\tilde\tau(\omega) := \int\tau(y)d\omega(y).$$

Assume there is another ergodic hyperbolic measure ω' with a Df-invariant splitting

$$T_yM = E_1(y)\bigoplus E_2(y)\bigoplus\cdots\bigoplus E_s(y), \quad \dim(E_r)=n_r, \quad 1\le r\le s, \quad y\in O(\omega'). \tag{2}$$

Under these assumptions we further describe the approximation of Oseledec splittings. Remember both (1) and (2) are arranged according to the increasing order of Lyapunov exponents.

Definition 1.6. Let $\eta>0$. The Oseledec splitting (1) of ω is η approximated by (2) of ω', if there exists a measurable subset Γ satisfying:

(a). $\omega(\Gamma)>1-\eta$;

(b). for any $x \in \Gamma$, there exist a point $z = z(x) \in \mathrm{supp}(\omega') \cap O(\omega')$ and $\beta(z) = (E_1(z), \cdots, E_s(z)) \in \mathcal{V}(n_1, \cdots, n_s; z)$ such that

$$\mathrm{dist}(\gamma, \beta) < \eta,$$

where the Oseledets bundle $\gamma(x) = (E_1(x), \cdots, E_s(x)) \in \mathcal{V}(n_1, \cdots, n_s; x)$.

Now we state our main result of this note.

Theorem 1.7. *Let $f : M \to M$ be a C^{1+r} diffeomorphism preserving an ergodic hyperbolic measure ω with its Oseledets splitting*

$$T_\Lambda M = E_1(\Lambda) \bigoplus E_2(\Lambda) \bigoplus \cdots \bigoplus E_s(\Lambda), \quad \dim(E_i) = n_i, \quad 1 \le i \le s$$

arranged according to the increasing order of Lyapunov exponents of ω, where $\Lambda = \bigcup_{k\ge 1} \Lambda_k$ is the Pesin set associated with ω. Given $\varepsilon > 0$, there is a hyperbolic periodic orbit $\mathrm{orb}(z,\, f)$ *together with an invariant splitting*

$$T_z M = E_1(z) \bigoplus E_2(z) \bigoplus \cdots \bigoplus E_s(z), \quad \dim(E_i) = n_i, \quad 1 \le i \le s$$

at z arranged according to the increasing order of Lyapunov exponents of the orbit $\mathrm{orb}(z)$ *such that the atomic measure ω_z supported on* $\mathrm{orb}(z, f)$ *satisfies the following properties:*

(i). *Mean distance of ω and of ω_z are $\varepsilon-$close, that is,*

$$|md_G(\omega)(E_i(\Lambda), E_j(\Lambda)) - md_G(\omega_z)(E_i(\mathrm{orb}(z)), E_j(\mathrm{orb}(z)))| < \varepsilon, \quad \forall\, 1 \le i \ne j \le s;$$

(ii). *Independence numbers of ω and of ω_z are $\varepsilon-$close, that is,*

$$|\tilde{\tau}(\omega) - \tilde{\tau}(\omega_z)| < \varepsilon;$$

(iii). *The Oseledec splitting of ω is $\varepsilon-$approximated by that of ω_z.*

2. Proofs

In the beginning, we will find an isolated ergodic measure on $\mathcal{W}(n_1, \cdots, n_s)$ covering ω, which is needful in the the following proofs.

Lemma 2.1. *There exists one and only one invariant measure $m \in \mathcal{M}_{inv}(\mathcal{W}, D^\# f)$ such that $\pi_*(m) = \omega$ and*

$$\int_{\mathcal{W}} \phi_i dm = \sum_{j=1}^{i} n_j \lambda_j, \quad \int_{\mathcal{W}} \psi_i dm = \sum_{j=i+1}^{s} n_j \lambda_j, \quad 1 \le i \le s-1.$$

In addition, m is ergodic and $m(\mathcal{W}_1) = 1$.

Proof. By Oseledets theorem, take and fix a point $x \in Q_m(M,\, f)$, $\gamma_0 = (E_1(x), \cdots, E_s(x)) \in \mathcal{V}$ so that

$$\lim_{n\to\pm\infty} \frac{1}{n} \Sigma_{i=0}^{n-1} \phi_t((Df^i \gamma_0) = n_t \lambda_t, \;\; 1 \le t \le s.$$

Let $\gamma_1 = \sigma(\gamma_0)$. Define a sequence of measures μ_n on $\mathcal{W}$ by

$$\int \phi d\mu_n := \frac{1}{n} \Sigma_{i=0}^{n-1} \phi(D^\# f^i \gamma_1), \quad \forall \phi \in C^0(\mathcal{W}, \mathbb{R}).$$

By taking a subsequence when necessary we can assume that $\mu_n \to \nu_0$. It is standard to verify that ν_0 is a $D^\# f$-invariant measure and ν_0 covers m, i.e., $\pi_*(\nu_0) = \omega$. Set

$$Q(\mathcal{W}, D^\# f) := \cup_{\nu \in \mathcal{M}_{erg}(\mathcal{W}, D^\# f)} Q_\nu(\mathcal{W}, D^\# f).$$

Then $Q(\mathcal{W}, D^\# f)$ is a $D^\# f-$invariant total measure subset in $\mathcal{W}$. We have

$$\begin{aligned} &\omega(Q_\omega(M,f) \cap \pi Q(\mathcal{W}, D^\# f)) \\ \geq &\nu_0(\pi^{-1} Q_\omega(M,f) \cap Q(\mathcal{W}, D^\# f)) \\ = &1. \end{aligned}$$

Then the set

$$\begin{aligned} \mathcal{A} := \{\mu \in \mathcal{M}_{erg}(\mathcal{W}, D^\# f) \mid \exists\, \gamma \in Q(\mathcal{W}, D^\# f), \pi(\gamma) \in Q_\omega(M,f), s.t. \\ \lim_{n\to+\infty} \frac{1}{n} \Sigma_{i=0}^{n-1} \phi(D^\# f^i \gamma) = \lim_{n\to-\infty} \frac{1}{n} \Sigma_{i=0}^{n-1} \phi(D^\# f^i \gamma) \\ = \int_{\mathcal{W}} \phi\, d\mu \qquad\qquad \forall \phi \in C^0(\mathcal{W}, \mathbb{R}) \,\} \end{aligned}$$

is non-empty. It is clear that μ covers ω, $\pi_*(\mu) = \omega$, for all $\mu \in \mathcal{A}$. And we claim that $\mathcal{A}$ coincides with the set of all the measures in $\mathcal{M}_{erg}(\mathcal{W}, D^\# f)$ that cover ω. In fact, if $\mu \in \mathcal{M}_{erg}(\mathcal{W}, D^\# f)$ covers ω, $\pi_*\mu = \omega$, from the fact that $\mu(Q_\mu(\mathcal{W}, D^\# f)) = 1$, we have

$$\begin{aligned} &\omega(Q_\omega(M,f) \cap \pi Q_\mu(\mathcal{W}, D^\# f)) \\ \geq &\mu(\pi^{-1} Q_\omega(M,f) \cap Q_\mu(\mathcal{W}, D^\# f)) \\ = &1. \end{aligned}$$

Thus there is $\beta \in Q_\mu(\mathcal{W}, D^\# f)$ with $\pi(\beta) \in Q_\omega(M,f)$, which means $\mu \in \mathcal{A}$. Therefore,

$$\mathcal{A} = \{\mu \in \mathcal{M}_{erg}(\mathcal{W}, D^\# f) \mid \pi_*(\mu) = \omega\}.$$

Assume the ergodic decomposition of ν_0 is of form

$$\nu_0 = \int_{\mathcal{M}_{erg}(\mathcal{W}, D^\# f)} d\tau_{\nu_0}(m).$$

Then $\tau_{\nu_0}(\mathcal{A}) = 1$. Since $\mu_n \to \nu_0$ and ϕ_i $(1 \leq i \leq s)$ are continuous,

$$\int_{\mathcal{W}} \phi_i d\nu_0 = \lim_{n\to+\infty} \int_{\mathcal{W}} \phi_i d\mu_n = \sum_{t=1}^{i} n_t \lambda_t.$$

Using the ergodic decomposition of ν_0, we obtain

$$\int_{\mathcal{M}_{erg}(\mathcal{W}, D^\# f)} \int_{\mathcal{W}} \phi_i dm d\tau_{\nu_0}(m) = \int_{\mathcal{W}} \phi_i d\nu_0 = \sum_{t=1}^{i} n_t \lambda_t, \quad 1 \leq i \leq s-1.$$

Observe that for any $m \in \mathcal{A}$, for m-almost γ

$$\int_{\mathcal{W}} \phi_i dm = \lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi_i(D^\# f^j \gamma) \geq \sum_{t=1}^{i} n_t \lambda_t,$$

and the equality holds if and only if $\pi_i(\gamma(x)) = \oplus_{t=1}^{i} E_t(x)$. Hence, for τ_{ν_0}-almost $m \in \mathcal{A}$, for m-almost $\gamma(x) \in \mathcal{W}$,

$$\int_{\mathcal{W}} \phi_i dm = \lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi_i(D^{\#} f^j \gamma) = \sum_{t=1}^{i} n_t \lambda_t, \ \ 1 \le i \le s-1.$$

In the same manner, for any $m \in \mathcal{A}$, for m-almost γ

$$\int_{\mathcal{W}} \psi_i dm = \lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \psi_i(D^{\#} f^j \gamma) \le \sum_{t=i+1}^{s} n_t \lambda_t,$$

and the equality holds if and only if $\widehat{\pi}_i(\gamma(x)) = \oplus_{t=i+1}^{s} E_t(x)$. It follows that for τ_{ν_0}-almost $m \in \mathcal{A}$, for m-almost $\gamma(x) \in \mathcal{W}$,

$$\int_{\mathcal{W}} \psi_i dm = \lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \psi_i(D^{\#} f^j \gamma) = \sum_{t=i+1}^{s} n_t \lambda_t, \ \ 1 \le i \le s-1.$$

Additionally, if $m' \in \mathcal{M}_{inv}(\mathcal{W}, D^{\#} f)$ satisfying that $\pi_*(m') = \omega$ and

$$\int_{\mathcal{W}} \phi_i dm = \sum_{j=1}^{i} n_j \lambda_j, \quad \int_{\mathcal{W}} \psi_i dm = \sum_{j=i+1}^{s} n_j \lambda_j, \quad 1 \le i \le s-1,$$

then by above proof we can see that for m'-almost $\gamma(x) \in \mathcal{W}$, $\pi_i(\gamma(x)) = \oplus_{t=1}^{i} E_t(x)$, $\widehat{\pi}_i(\gamma(x)) = \oplus_{t=i+1}^{s} E_t(x)$, $1 \le i \le s-1$. So, $m' = m$. That is, m is unique and ergodic and $m(\mathcal{W}_1) = 1$. $\square$

Proof of Theorem 1.7

Take a decreasing sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ which approaches zero. We further assume that

$$\varepsilon_1 < \frac{1}{2} \min \big\{ |\lambda_i - \lambda_j| \mid 1 \le i \ne j \le s \big\}.$$

Applying Theorem 1.3 to the ergodic measure ω, we can find a hyperbolic periodic point z_n with period p_n which satisfies the following properties:

(a) the atomic measure $\omega_n = \frac{1}{p_n} \sum_{i=1}^{p_n-1} \delta_{f^i z_n}$ is ε_n−close to the measure ω in the weak*-topology.

(b) Without loss of generality we assume that ω_n has an invariant splittings $E_1^n \oplus E_2^n \oplus \cdots \oplus E_s^n$ on $T_{\mathrm{orb}(z_n,f)} M$ such that $\dim(E_i^n) = n_i$ $(1 \le i \le s)$ and for any $0 \ne v \in E_i^n(z_n)$, the Lyapunov exponents

$$\lambda_i - \varepsilon_n < \liminf_{k\to\infty} \frac{1}{k} \log \|Df_{z_n}^k v\| \le \limsup_{k\to\infty} \frac{1}{k} \log \|Df_{z_n}^k v\| < \lambda_i + \varepsilon_n. \tag{3}$$

Let $\xi_n(x) = (E_1^n(x), E_2^n(x), \cdots E_s^n(x))$ for $x \in \mathrm{orb}(z)$. Thus, there is an ergodic invariant measure $m_n \in \mathcal{M}_{erg}(\mathcal{W}_1, D^{\#} f)$ such that $\pi_*(m_n) = \omega_n$ and

$$m_n(\sigma(\xi_n(f^t z_n))) = \frac{1}{p_n}, \quad 0 \le t \le p_n - 1.$$

At most taking a subsequence, we suppose that m_n converges to an invariant $\mu \in \mathcal{M}_{inv}(\mathcal{W}, D^{\#} f)$. Since $\pi_*(m_n) = \omega_n$ and $\lim_{n\to+\infty} \omega_n = \omega$, so $\pi_*(\mu) = \omega$.

By (3), it is easy to verify that

$$\sum_{j=1}^{i} n_j(\lambda_j - \varepsilon_n) \quad < \quad \int_{\mathcal{W}} \phi_i dm_n < \sum_{j=1}^{i} n_j(\lambda_j + \varepsilon_n), \quad 1 \le i \le s-1,$$

$$\sum_{j=i+1}^{s} n_j(\lambda_j - \varepsilon_n) \quad < \quad \int_{\mathcal{W}} \psi_i dm_n < \sum_{j=i+1}^{s} n_j(\lambda_j + \varepsilon_n), \quad 1 \le i \le s-1.$$

Letting $n \to \infty$, we deduce

$$\int_{\mathcal{W}} \phi_i d\mu \quad = \quad \sum_{j=1}^{i} n_j \lambda_j, \quad 1 \le i \le s-1,$$

$$\int_{\mathcal{W}} \psi_i d\mu \quad = \quad \sum_{j=i+1}^{s} n_j \lambda_j, \quad 1 \le i \le s-1.$$

By Lemma 2.1, it follows that $\mu = m$.

Denote $\rho_0 = \sigma_*^{-1}(m)$, $\rho_n = \sigma_*^{-1}(m_n)$. Then $\rho_0, \rho_n \in \mathcal{M}_{erg}(\mathcal{V}, D^{\#}f)$ and $\rho_n \to \rho_0$ as $n \to +\infty$.

Proof of (i) We need verify that

$$(4)\ \lim_{n\to\infty} \frac{1}{p_n} \sum_{k=0}^{p_n-1} \angle(E_i^n(f^k(z_n)),\, E_j^n(f^k(z_n))) = m\angle_\omega(E_i,\, E_j),\ 1 \le i \ne j \le s,$$

where $E_1^n(\mathrm{orb}(z)) \oplus \cdots \oplus E_s^n(\mathrm{orb}(z))$ is the invariant splitting given as above (b). We define $\phi_{ij} : \mathcal{U} \to \mathbb{R}$, $(E_1, \cdots, E_s) \to \angle(E_i, E_j)$. Then ϕ_{ij} is bounded and continuous. Noting that $\rho_n \to \rho_0$, we get (4).

Proof of (ii) We recall the measures ω_n and ω and m_n and m and their relations:

$$\omega_n \to \omega, \quad \rho_n \to \rho_0, \quad \pi_*(\rho_n) = \omega_n, \quad \pi_*(\rho_0) = \omega.$$

Since the function $\tau : \mathcal{U} \to \mathbb{R}$ is continuous, so

$$\int \tau\, d\rho_n \to \int \tau\, d\rho_0.$$

Moreover, the ergodic measure $m \in \mathcal{M}_{erg}(\mathcal{U}, D^{\#}f)$ is unique in the sense of Lemma 2.1, which implies

$$\int \tau\, d\rho_0 = \int \tau\, d\omega.$$

Note that ω_n is an ergodic measure whose spectrum are increasingly arranged as $E_1^n \oplus \cdots \oplus E_s^n$. This implies that

$$\int \tau\, d\rho_n = \int \tau\, d\omega_n.$$

Hence we have that $\int \tau\, d\omega_n \to \int \tau\, d\omega$ or $\tilde{\tau}(\omega_n) \to \tilde{\tau}(\omega)$. We thus obtain (ii).

Proof of (iii) Given $\varepsilon > 0$, take l large, so that

$$\omega(\Lambda_l(\omega)) > 1 - \varepsilon, \tag{4.12}$$

where $\Lambda_l(\omega)$ denotes the $l-th$ Pesin block associated with ω. Since the splitting

$$x \to E_1(x) \oplus \cdots \oplus E_s(x)$$

depends continuously on $x \in \Lambda_l(\omega)$, we can choose a uniform constant $\eta > 0$ satisfying

$$\eta < \min_{\substack{i \neq j \\ x \in \Lambda_l(\tilde{\omega})}} \{\frac{1}{10} \angle(E_i(x),\, E_j(x)),\, \varepsilon\}. \tag{4.13}$$

For each $x \in \Lambda_l(\omega) \cap \mathrm{supp}(\omega)$, we take and fix $\alpha_0(x) = (E_1(x), \cdots, E_s(x))$. Denote by $B(\alpha_0(x),\, \eta)$ the $\eta-$neighborhood of $\alpha_0(x)$ under the metric on the Grassman bundle. Then $m(B(\alpha_0(x),\, \eta)) > 0$. Recalling that $\rho_n \to \rho_0$, we have

$$\liminf_{n \to +\infty} \rho_n(B(\alpha_0(x),\, \eta)) \geq \rho_0(B(\alpha_0(x),\, \eta)) > 0.$$

Therefore, we can take an integer $N(x) = N(\alpha_0(x)) > 0$ such that

$$\rho_n(B(\alpha_0(x),\, \eta)) > 0, \quad \forall\, n \geq N(x).$$

This implies the existence of an element $\beta_n(x)$ in $Q_{\rho_n}(\mathcal{U}, D^{\#}f) \cap \mathrm{supp}(\rho_n) \cap B(\alpha_0(x),\, \eta)$ for each $n \geq N(x)$. Observing that ρ_n covers the atomic measure ω_n, we can deduce that $\beta_n(x)$ must be an element based on a periodic point $z(x, n)$ on $\mathrm{orb}(z_n)$, where z_n is the periodic point chosen in $(a)(b)$. By the uniqueness of ρ_n, we know that $\beta_n(x) = (E_1^n(z(x,n)), \cdots, E_s^n(z(x,n))$. Thus the Oseledets splitting of ω at x is η approximated by the invariant splitting $E_1^n(z(x,n)) \oplus \cdots \oplus E_s^n(z(x,n)$ of ω_n at a point $z = z(x,\, n)$ on $\mathrm{orb}(z_n),\ n \geq N(x)$. Note that the number $N(x)$ may vary with x, we need to find a number N_1, independent of the choice of $x \in \Lambda_l(\omega)$, such that ρ_{N_1} meets (iii). This can be done by the compactness of $\Lambda_l(\omega)$ and continuity of the Oseledets splitting on it.

Hence we complete the proof of Theorem 1.7. □

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